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Differential calculus on the quantum Heisenberg group

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Abstract. The differential calculus on the quantum Heisenberg group is constructed. The duality between the quantum Heisenberg group and algebra is proved. It is shown that the calculus considered can be obtained from 3D calculus on $SU_\mu(2)$ by contraction.

1. Introduction

The one-dimensional deformed Heisenberg group and algebra were investigated in [1, 2]. In this paper, using Woronowicz's theory [3], we construct the bicovariant differential calculus on the deformed one-dimensional Heisenberg group and we describe the structure of its quantum Lie algebra. Then we prove that our quantum Lie algebra is equivalent to the one-dimensional deformed Heisenberg algebra. We also show that our calculus can be obtained by contraction from Woronowicz's 3D calculus on $SU_\mu(2)$ in spite of the fact that the latter is not bicovariant.

2. The differential calculus

The quantum group $H(1)_q$ is a matrix quantum group à la Woronowicz [4]

$$T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

where the matrix elements α, β, δ generate the algebra $\mathcal{A}$ and satisfy the following relations [1]:

$$\begin{aligned} [\alpha, \beta] &= \mathrm{i}\lambda\alpha \\ [\delta, \beta] &= \mathrm{i}\lambda\delta \\ [\alpha, \delta] &= 0 \end{aligned} \tag{2}$$

λ being a real parameter.

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The co-product, co-unit and antipode are given by

$$\begin{aligned}
&\Delta(\alpha) = I \otimes \alpha + \alpha \otimes I \\
&\Delta(\beta) = I \otimes \beta + \beta \otimes I + \alpha \otimes \delta \\
&\Delta(\delta) = I \otimes \delta + \delta \otimes I \\
&S(\alpha) = -\alpha \\
&S(\beta) = -\beta + \alpha\delta \\
&S(\delta) = -\delta \\
&\varepsilon(\alpha) = \varepsilon(\beta) = \varepsilon(\delta) = 0.
\end{aligned} \tag{3}$$

The main ingredient of the Woronowicz theory is the choice of a right ideal in $\ker\varepsilon$, which is invariant under the adjoint action of the group. The adjoint action is defined as follows:

$$\mathrm{ad}(a) = \sum_k b_k \otimes S(a_k)c_k \tag{4}$$

where

$$(\Delta \otimes I) \circ \Delta(a) = \sum_k a_k \otimes b_k \otimes c_k.$$

One can prove the following:

Theorem 1. Let $\mathcal{R} \subset \ker\varepsilon$ be the right ideal generated by the following elements: α^2, δ^2, $\beta\alpha$, $\beta\delta$, $\alpha\delta$, $\beta^2 - 2\mathrm{i}\lambda\beta$. Then

(i) $\mathcal{R}$ is ad-invariant, $\mathrm{ad}(\mathcal{R}) \subset \mathcal{R} \otimes \mathcal{A}$;

(ii) $\ker\varepsilon/\mathcal{R}$ is spanned by the following elements: α, β, δ.

Having established the structure of $\mathcal{R}$ we follow closely the Woronowicz construction. The basis of the space of the left-invariant 1-forms consists of the following elements:

$$\begin{aligned}
&\omega_\alpha \equiv \pi r^{-1}(I \otimes \alpha) = \mathrm{d}\alpha \\
&\omega_\beta \equiv \pi r^{-1}(I \otimes \beta) = \mathrm{d}\beta - \alpha\, \mathrm{d}\delta \\
&\omega_\delta \equiv \pi r^{-1}(I \otimes \delta) = \mathrm{d}\delta
\end{aligned} \tag{5}$$

where the mapping r^{-1} is given by

$$r^{-1}(a \otimes b) = (a \otimes I)(S \otimes I)\Delta(b) \qquad a, b \in \mathcal{A}$$

and the mapping π is given by

$$\pi\Big(\sum_k a_k \otimes b_k\Big) = \sum_k a_k\, \mathrm{d}b_k$$

where $\sum_k a_k \otimes b_k \in \mathcal{A} \otimes \mathcal{A}$ is an element such that

$$\sum_k a_k b_k = 0.$$

The next step is to find the commutation rules between the invariant forms and generators of $\mathcal{A}$. The detailed calculations result in the following formulae:

$$\begin{aligned}
&[\alpha, \omega_\alpha] = 0\\
&[\delta, \omega_\alpha] = 0\\
&[\beta, \omega_\alpha] = -\mathrm{i}\lambda\omega_\alpha\\
&[\alpha, \omega_\delta] = 0\\
&[\delta, \omega_\delta] = 0\\
&[\beta, \omega_\delta] = -\mathrm{i}\lambda\omega_\delta\\
&[\alpha, \omega_\beta] = 0\\
&[\delta, \omega_\beta] = 0\\
&[\beta, \omega_\beta] = -2\mathrm{i}\lambda\omega_\beta.
\end{aligned} \tag{6}$$

Then, following Woronowicz's paper [3], we can construct the right-invariant forms

$$\begin{aligned}
&\eta_\alpha = \omega_\alpha\\
&\eta_\delta = \omega_\delta\\
&\eta_\beta = \omega_\beta - \omega_\alpha\delta + \omega_\delta\alpha.
\end{aligned} \tag{7}$$

This concludes the description of the bimodule Γ of 1-forms on $H(1)_q$. The external algebra can now be constructed as follows [3]. On $\Gamma^{\otimes 2}$ we define a bimodule homomorphism σ such that

$$\sigma(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega \tag{8}$$

for any left-invariant $\omega \in \Gamma$ and any right-invariant $\eta \in \Gamma$. Then by definition

$$\Gamma^{\wedge 2} = \frac{\Gamma^{\otimes 2}}{\ker(I - \sigma)}. \tag{9}$$

Equations (7)–(9) allow us to calculate the external product of left-invariant 1-forms. The result reads

$$\begin{aligned}
&\omega_\beta \wedge \omega_\alpha = -\omega_\alpha \wedge \omega_\beta\\
&\omega_\beta \wedge \omega_\delta = -\omega_\delta \wedge \omega_\beta\\
&\omega_\beta \wedge \omega_\beta = 0\\
&\omega_\alpha \wedge \omega_\alpha = 0\\
&\omega_\delta \wedge \omega_\delta = 0\\
&\omega_\alpha \wedge \omega_\delta = -\omega_\delta \wedge \omega_\alpha.
\end{aligned} \tag{10}$$

To complete the external calculus, we derive the Cartan–Maurer equations

$$\begin{aligned}
&\mathrm{d}\omega_\alpha = 0\\
&\mathrm{d}\omega_\delta = 0\\
&\mathrm{d}\omega_\beta = -\omega_\alpha \wedge \omega_\delta.
\end{aligned} \tag{11}$$

3. Quantum Lie algebra

In order to obtain the counterpart of the classical Lie algebra, we introduce the counterpart of the left-invariant vector fields. They are defined by the formula

$$\mathrm{d}a = (\chi_\alpha * a)\omega_\alpha + (\chi_\beta * a)\omega_\beta + (\chi_\delta * a)\omega_\delta. \tag{12}$$

In order to find the quantum Lie algebra, we apply the external derivative to both sides of (12). We use $d^2 a = 0$ on the left-hand side and calculate the right-hand side using (11) and again (12). Nullifying the coefficients in front of basis elements of $\Gamma^{\wedge 2}$, we find the quantum Lie algebra

$$\begin{aligned} &[\chi_\alpha, \chi_\beta] = 0 \\ &[\chi_\delta, \chi_\beta] = 0 \\ &[\chi_\alpha \chi_\delta] = \chi_\beta. \end{aligned} \tag{13}$$

From the Woronowicz theory, it follows that the co-product of the functional φ_i ($\varphi_i \equiv \chi_\alpha, \chi_\beta, \chi_\delta$) can be written in the form

$$\Delta\varphi_i = \sum_j \varphi_j \otimes f_{ji} + I \otimes \varphi_i \tag{14}$$

where f_{ji} are the functionals entering in the commutation rules between the left-invariant forms and elements of $\mathcal{A}$

$$\omega_j a = \sum_i (f_{ji} * a)\omega_i. \tag{15}$$

Then, it follows from commutation rules (6) that the co-product for our functionals can be written in the following form:

$$\begin{aligned} \Delta\chi_\alpha &= \chi_\alpha \otimes f_\alpha + I \otimes \chi_\alpha \\ \Delta\chi_\beta &= \chi_\beta \otimes f_\beta + I \otimes \chi_\beta \\ \Delta\chi_\delta &= \chi_\delta \otimes f_\delta + I \otimes \chi_\delta. \end{aligned} \tag{16}$$

Using the fact that $\Delta f_i = f_i \otimes f_i$ ($i = \alpha, \beta, \delta$) [3] and (6) and (15), we can calculate the functionals f_i. After some calculations we obtain

$$\begin{aligned} f_\alpha &= (I - 2i\lambda\chi_\beta)^{\frac{1}{2}} \\ f_\beta &= I - 2i\lambda\chi_\beta \\ f_\delta &= (I - 2i\lambda\chi_\beta)^{\frac{1}{2}}. \end{aligned} \tag{17}$$

Now it is easy to see that the substitution

$$\begin{aligned} \chi_\delta &= B_0 \\ \chi_\beta &= B_1 \\ \chi_\alpha &= B_2 \end{aligned} \tag{18}$$

reproduces the structure of the Hopf algebra generated by the infinitesimal generators (obtained by contraction procedure) of the quantum matrix pseudogroup $H(1)_q$, which was described in [1]. This proves the duality between the quantum Heisenberg group and algebra.

4. Contraction from $SU_\mu(2)$†

We shall show how our calculus can be obtained by contraction from 3D left-covariant calculus on $SU_\mu(2)$ constructed by Woronowicz ([5]). The $SU_\mu(2)$ quantum group is a matrix quantum group

$$X = \begin{pmatrix} \sigma & -\mu\rho^* \\ \rho & \sigma^* \end{pmatrix} \tag{19}$$

† The content of this section was suggested by one of the referees. We are grateful for his remarks.

where μ is a deformation parameter while the generators obey the following algebraic rules

$$\begin{aligned}
&\sigma^*\sigma + \rho^*\rho = I \\
&\sigma\sigma^* + \mu^2\rho\rho^* = I \\
&\rho\rho^* = \rho^*\rho \\
&\mu\rho\sigma = \sigma\rho \\
&\mu\rho^*\sigma = \sigma\rho^*.
\end{aligned} \tag{20}$$

The above rules can be solved by putting

$$\sigma = \mathrm{e}^{\mathrm{i}\varphi}\tau \tag{21}$$

and imposing

$$\begin{aligned}
&\tau^2 + \rho^*\rho = I \\
&\mathrm{e}^{\mathrm{i}\varphi}\rho = \mu\rho\mathrm{e}^{\mathrm{i}\varphi} \\
&\tau^* = \tau \qquad \varphi^* = \varphi.
\end{aligned} \tag{22}$$

From $SU_\mu(2)$ we can obtain by contraction the Heisenberg group. To this end we define new variables α, β, δ by

$$\begin{aligned}
&\rho = \frac{\delta + \mathrm{i}\alpha}{\sqrt{2}R} \qquad \rho^* = \frac{\delta - \mathrm{i}\alpha}{\sqrt{2}R} \\
&\varphi = \frac{1}{R^2}\left(\beta - \frac{\alpha\delta}{2}\right) \qquad \mu = \mathrm{e}^{\lambda/R^2}
\end{aligned} \tag{23}$$

and put $R \to \infty$. It is easy to check that as a result the Heisenberg group emerges.

Let us consider the 3D calculus on $SU_\mu(2)$ described in [5]. The left-invariant forms ω_0, ω_1 and ω_2 are defined as follows:

$$\begin{aligned}
&\omega_0 = \rho^*\,\mathrm{d}\sigma^* - \mu\sigma^*\,\mathrm{d}\rho^* \\
&\omega_1 = \sigma^*\,\mathrm{d}\sigma + \rho^*\,\mathrm{d}\rho \\
&\omega_2 = \rho\,\mathrm{d}\sigma - \mu^{-1}\sigma\,\mathrm{d}\rho.
\end{aligned} \tag{24}$$

It is straightforward to check that

$$\begin{aligned}
&\lim_{R\to\infty} R\omega_0 = \frac{\mathrm{i}\,\mathrm{d}\alpha - \mathrm{d}\delta}{\sqrt{2}} \\
&\lim_{R\to\infty} R\omega_2 = \frac{\mathrm{i}\,\mathrm{d}\alpha + \mathrm{d}\delta}{\sqrt{2}} \\
&\lim_{R\to\infty} R^2\omega_1 = \mathrm{i}(\mathrm{d}\beta - \alpha\,\mathrm{d}\delta).
\end{aligned} \tag{25}$$

Moreover, under this limit the whole structure described in [5] is transformed into the formulae given in section 2.

Although the above reasoning shows that our calculus results from the contraction of the Woronowicz one it seems to be interesting to find also our ideal $\mathcal{R}$ (theorem 1) from that of Woronowicz. The latter is generated by the following six elements:

$$\begin{aligned}
&\rho^2 \qquad \rho^{*2} \qquad \rho\rho^* \\
&(\sigma - I)\rho \qquad (\sigma - I)\rho^* \\
&\sigma^* + \mu^2\sigma - (1+\mu^2)I.
\end{aligned} \tag{26}$$

First, by taking the limit $R \to \infty$ of $R^2(\rho^2, \rho^{*2}, \rho\rho^*)$ we infer that α^2, δ^2 and $\alpha\delta$ belong to $\mathcal{R}$.

Let us further note that σ has the following expansion

$$\sigma = I + \frac{\mathrm{i}}{R^2}\left(\beta - \frac{\alpha\delta}{2}\right) - \frac{\delta^2 + \alpha^2}{4R^2} + \mathrm{O}\left(\frac{1}{R^4}\right). \tag{27}$$

From (27) we conclude by taking the limits $\lim_{R\to\infty} R^3(\sigma - I)\rho$, $\lim_{R\to\infty} R^3(\sigma - I)\rho^*$ and skipping terms which are already known to belong to $\mathcal{R}$, that $\beta\alpha \in \mathcal{R}$, $\beta\delta \in \mathcal{R}$. Finally, let us note that, due to $\rho\rho^* \in \mathcal{R}$,

$$\sigma = \mathrm{e}^{\mathrm{i}\varphi}\sqrt{1 - \rho\rho^*} \underset{\mathcal{R}}{\simeq} \mathrm{e}^{\mathrm{i}\varphi}. \tag{28}$$

We can now use (28) and expand the last generator (26) up to the order $1/R^4$ to find that $\beta^2 - 2\mathrm{i}\lambda\beta \in \mathcal{R}$ (again neglecting terms which have already been shown to belong to $\mathcal{R}$).

Let us note that the similar contraction procedure can be applied to $4\mathrm{D}_+$ calculus on $SU_\mu(2)$ [6] in order to obtain a bicovariant calculus on deformed $E(2)$ described in [7].

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